A Small Dual-Frequency Transformer in Two Sections

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***Abstract*—One of the most useful transmission-line constructs is the quarter-wave transformer that is used to impedance match a line at a single frequency f_0. The feasibility of an electrically small transformer with two sections and capable of achieving ideal impedance matching at two arbitrary frequencies is demonstrated analytically. To achieve this, the exact solution to the resulting transcendental transmission-line equations for two sections is obtained with no restrictions. The parameters of the transformer are presented in explicit closed form, and are exact. The results of this study are useful for a number of practical design problems, including dual-band antennas and RF circuits in general. In particular, feasibility of ideal operation at the important first harmonic frequency $2f_0$ is demonstrated.**

***Index Terms*—Dual frequency, impedance matching, transformer, transmission lines.**

I. Introduction

QUARTER-WAVE transformers are ubiquitous in engineering and physics. Known also as quarter-wave plates and quarter-wave sections, they are the most popular kind of impedance transformers. Impedance transformers encompass lumped networks, tapered sections, shorting plungers, and double-stub matching [1], as well as a lesser known class, the so-called transmission-line transformer (TLT) composed of transmission lines with twisted connections [2]. Impedance transformers have traditionally been broadly divided into two groups; those with a continuously tapered impedance distribution and those with a stepped piece-wise impedance distribution. The latter being considerably shorter than the broad-band tapered transformers perhaps because they tend to mimic a traditional lumped-element design.

In a very recent publication [3], a new line impedance transformer has been presented. Chow and Wan introduce a dual-band two-section 1/3-wavelength transformer length that operates at the fundamental frequency f_1 and its first harmonic $2f_1$. Such a novel transformer is of great interest because of the current trend toward compact, smaller, and more efficient RF front ends, and exploitation of frequency reuse in commercial and military systems.

The transformer of Chow and Wan was designed by numerical solution of the transcendental equations obtained by enforcing operation at f_1 and $2f_1$ (four equations in four unknowns, line and impedance for each line section, all real quantities). In addition, a design equation was presented ([3, eq. (2)]), which, although not analytically exact, was numerically *nearly exact*. It was further stated in [3] that the analytical software *Mathematica*[1] was used to prove that the two-section transformer was not exact, but for engineering applications "effectively exact" for impedance transform ratios K not sufficiently high (K up to 6 or 15 depending on the tolerance on the resulting reflection coefficient levels). Chow and Wan further rationalize that the inexact nature of the dual-band two-section transformer was the reason why it was not discovered before from an exact transmission-line analysis.

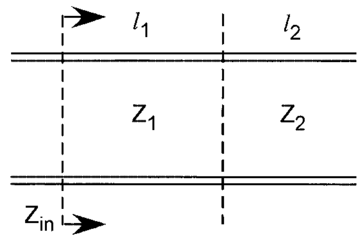

Fig. 1. Two-section dual-band transformer.

Here, we present an extension of the two-section transformer analysis of [3] to any two arbitrary frequencies f_1 and f_2, and obtain an exact analytical solution to the resulting transcendental equations, leading to practical design equations, which are validated numerically. Analytically and numerically, we demonstrate that the two-section transformer provides a true exact solution to the dual-frequency problem, under all frequency and loading conditions. The present solution, when applied to the case of fundamental plus first harmonic (f_1 and $f_2 = 2f_1$), results in an improvement on the analysis of [3], as our solution is analytically and numerically demonstrated to be valid under all impedance transform ratios.

II. Analysis

The input impedance $Z_{\rm in}$ of the two-section line shown in Fig. 1 is given by

$$Z_{\rm in} = Z_1 \frac{Z_L' + jZ_1 \tan(\beta\ell_1)}{Z_1 + jZ_L' \tan(\beta\ell_1)} \quad (1)$$

$$Z_L' = Z_2 \frac{R_L + jZ_2 \tan(\beta\ell_2)}{Z_2 + jR_L \tan(\beta\ell_2)}. \quad (2)$$

We want the input impedance to be equal to Z_0 at the two frequencies of interest f_1 and f_2. Equating $Z_{\rm in}$ to Z_0 and solving for Z_L' from (1) leads to

$$Z_L' = Z_1 \frac{Z_0 - jZ_1 \tan(\beta\ell_1)}{Z_1 - jZ_0 \tan(\beta\ell_1)}. \quad (3)$$

[1]Wolfram Res. Inc., Champaign, IL.

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This is equivalent to looking toward the left-hand side from the central transition plane. From (2) and (3), we eliminate leading to the following two identities:

$$\left(Z_1^2 R_L - Z_2^2 Z_0\right)\tan(\beta\ell_1)\tan(\beta\ell_2) = Z_1 Z_2 (R_L - Z_0) \quad (4)$$

$$Z_1\left(Z_2^2 - R_L Z_0\right)\tan(\beta\ell_2) = Z_2\left(R_L Z_0 - Z_1^2\right) \times \tan(\beta\ell_1). \quad (5)$$

These have been obtained by separating the real and imaginary components and assuming all parameters are real.

Equations (4) and (5) can be rewritten as follows:

$$\tan(\beta\ell_1)\tan(\beta\ell_2) = \alpha \equiv \frac{Z_1 Z_2 (R_L - Z_0)}{(Z_1^2 R_L - Z_2^2 Z_0)} \quad (6)$$

$$\frac{\tan(\beta\ell_1)}{\tan(\beta\ell_2)} = \gamma \equiv \frac{Z_1 (Z_2^2 - R_L Z_0)}{Z_2 (R_L Z_0 - Z_1^2)} \quad (7)$$

which define the parameters α and γ.

From (6) and (7), we obtain the following identities:

$$(\tan(\beta\ell_1))^2 = \alpha\gamma \quad (8)$$

$$(\tan(\beta\ell_2))^2 = \frac{\alpha}{\gamma}. \quad (9)$$

Equations (8) and (9), when applied to frequencies f_1 and f_2, result in the following four transcendental equations:

$$(\tan(\beta_1\ell_1))^2 = \alpha\gamma \quad (10a)$$

$$(\tan(\beta_2\ell_1))^2 = \alpha\gamma \quad (10b)$$

$$(\tan(\beta_1\ell_2))^2 = \frac{\alpha}{\gamma} \quad (10c)$$

$$(\tan(\beta_2\ell_2))^2 = \frac{\alpha}{\gamma}. \quad (10d)$$

From (10a) and (10b), we obtain

$$\tan(\beta_2\ell_1) = \pm\tan(\beta_1\ell_1). \quad (11)$$

Whose solution is given by

$$\beta_2\ell_1 \mp \beta_1\ell_1 = n\pi \quad (12)$$

for n an arbitrary integer. Similarly, from (10c) and (10d), we obtain

$$\beta_2\ell_2 \mp \beta_1\ell_2 = m\pi \quad (13)$$

for m another arbitrary integer.

Since we are interested in a small transformer, we should pick the $(+)$ sign in (12) and (13), together with $m = n = 1$ (here, we assume $f_2 \geq f_1$), and we obtain as solution to the above transcendental equations as follows:

$$\ell_1 = \ell_2 \quad (14a)$$

$$\ell_2 = \frac{\pi}{\beta_2 + \beta_1}. \quad (14b)$$

With the line lengths known, α and γ can be determined through application of (6) and (7) at either f_1 or f_2. For instance, using f_1, and simplifying through the use of (14), we obtain

$$\alpha = \left(\tan(\beta_1\ell_1)\right)^2 \quad (15)$$

$$\gamma = 1. \quad (16)$$

Using $\gamma = 1$ in (7) results in

$$Z_0 R_L = Z_1 Z_2 \quad (17)$$

which is essentially the so-called antimetry condition [8]. Equation (17) is valid provided $(Z_1 + Z_2) \neq 0$. Similarly, after using (17) in (6), it becomes

$$Z_1^2 R_L - Z_2^2 Z_0 = \frac{Z_0 R_L (R_L - Z_0)}{\alpha}. \quad (18)$$

By using (17) in (18), a fourth-order equation for either Z_1 or Z_2 results. It is, however, a simple algebraic equation of the second order on the square of either Z_1 or Z_2. Solving for Z_1 by standard means, the only solution leading to real line impedance values is given by

$$Z_1 = \sqrt{\frac{Z_0}{2\alpha}(R_L - Z_0) + \sqrt{\left[\frac{Z_0}{2\alpha}(R_L - Z_0)\right]^2 + Z_0^3 R_L}} \quad (19)$$

and Z_2 can be obtained from (17) as

$$Z_2 = \frac{Z_0 R_L}{Z_1}. \quad (20)$$

This completes the derivation of the design equations for the two-section transformer.

III. Nature of the Solution

The solution, as presented through (14), (19), and (20), is elementary in nature, explicit, and in closed form; however, it is instructive to consider a few fine points.

A. Total Length of Transformer

This can be obtained from (14). The total length in wavelengths (in the material) calculated at the fundamental frequency f_1 is given by

$$\frac{\ell_1 + \ell_2}{\lambda_1} = \frac{\beta_1}{\beta_2 + \beta_1}. \quad (21)$$

In view of the fact that we assumed $f_2 \geq f_1$, it is seen that the total length will, in general, be smaller than one-half wavelength at the low-frequency end. This extreme is observed when $f_2 = f_1$, i.e., when we have a double pole. Since the equations do not smoothly consider this scenario, and some special limiting action needs to be taken, this is done elsewhere in this paper.

On the other hand, the equations indicate that the total length can be extremely small, and this can happen if $f_2 \gg f_1$, i.e., when a simple pole is encountered. This special case is treated separately elsewhere.

B. Single-Pole Limit

For $f_2 \gg f_1$, the total length is exceedingly small, and (15) becomes

$$\alpha \approx (\beta_1\ell_1)^2. \quad (22)$$

Further, this implies that from (19) and (20)

$$Z_1 \approx \frac{\sqrt{Z_0 |R_L - Z_0|}}{\beta_1\ell_1} \quad (23)$$

$$Z_2 \approx \beta_1 \ell_1 R_L \sqrt{\frac{Z_0}{|R_L - Z_0|}}. \tag{24}$$

Since, on the other hand [5], the wave impedance Z and wave velocity "v" are related to the distributed (per unit length) inductance L and capacitance C of a given line, we have

$$Z = \sqrt{\frac{L}{C}} \quad v = \frac{1}{\sqrt{LC}}. \tag{25}$$

Since the velocity is the ratio from angular frequency ω to wavenumber β, we evaluate this at f_1, replace in (25), and solve (25) for C and L. We obtain

$$L = \frac{Z\beta_1}{\omega_1} \quad C = \frac{\beta_1}{Z\omega_1}. \tag{26}$$

For exceedingly small lengths, (23) and (24) indicate that Z_1 is very large, while Z_2 is very small. Equation (25) means that Z_1 is eminently inductive, while Z_2 is eminently capacitive. It is thus permissible to replace the first section with a series lumped inductance equal to $L\ell_1$ (here, Z in (26) is Z_1), and the second section with a shunt lumped capacitance given by $C\ell_1$ (here, Z in (26) is Z_2). After some rearrangement, we find

$$L\ell_1 = \frac{\sqrt{Z_0 |R_L - Z_0|}}{\omega_1} \tag{27a}$$

$$C\ell_2 = \frac{1}{\omega_1 R_L} \sqrt{\frac{|R_L - Z_0|}{Z_0}}. \tag{27b}$$

The above combination of a lumped (series) inductance and (shunt) capacitance is actually a known transforming network using lumped reactive elements [6], and it is known as a ladder-type half-section. This validates the extension of the solution to the limiting case, and illustrates its roots.

Thus, the single-pole limit of our two-section transformer is an extension of a lumped impedance-transforming network. This is a fundamental difference between the present solution and that afforded by a quarter-wave transformer.

C. Double-Pole Limit

Since according to (21), when $f_2 = f_1$, each section becomes one-quarter wavelength, we are interested in elucidating the nature of this special case. From (15), we see that $\alpha \to \infty$ and, in this limit, (19) and (20) reduce to

$$Z_1 = Z_0^{3/4} R_L^{1/4} \tag{28a}$$

$$Z_2 = R_L^{3/4} Z_0^{1/4}. \tag{28b}$$

These equations are, however, popular equations for the design of two quarter-wave transformers in series [4], [7]. It is thus observed that the limits of the present solution are well-known designs of quite different nature, while the present design bridges the gap between them, and extends them considerably, since the limiting cases are single-frequency designs.

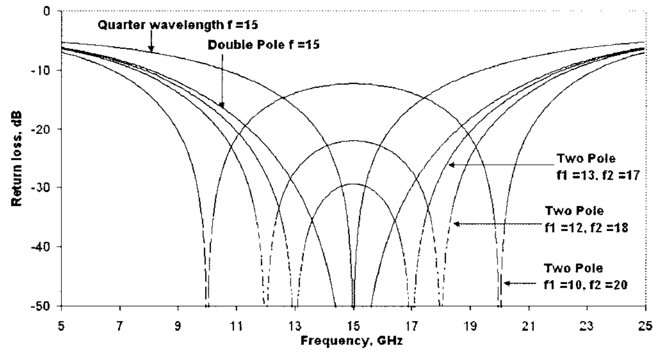

Fig. 2. Comparison of $|S11|$ for typical quarter-wavelength versus two-pole $(K = 4)$ response.

D. Symmetry Properties

The dimensionless impedance transform ratio parameter K is defined as [3]

$$K = \frac{Z_0}{R_L}. \tag{29}$$

With this, (19) becomes

$$\frac{Z_1}{R_L} = \sqrt{\frac{K}{2\alpha}(1-K) + \sqrt{\left[\frac{K}{2\alpha}(1-K)\right]^2 + K^3}} \tag{30}$$

and it can be shown by purely algebraic manipulations using (20) and (30) that the solution for K is related to the solution for $1/K$ via

$$\left.\frac{Z_1}{R_L}\right|_{K=K_0} = \left.\frac{Z_2}{Z_0}\right|_{K=1/K_0} \tag{31a}$$

$$\left.\frac{Z_2}{R_L}\right|_{K=K_0} = \left.\frac{Z_1}{Z_0}\right|_{K=1/K_0}. \tag{31b}$$

IV. CALCULATIONS

A few calculations have been made via direct computation of (14), (19), (20), and standard transmission line (1) and (2) to define the return loss S_{11} with respect to the line impedance Z_0. The first example corresponds to a comparison of the performances of a typical quarter-wavelength transformer versus two-pole responses. One of the four two-pole cases considered in the return-loss calculation of Fig. 2 is the double-pole case, which was shown analytically to be identical to the standard two quarter-wavelength section design. We have used a central frequency of 15 GHz, $\mathrm{RL} = 50\,\Omega$, and $Z_0 = 200\,\Omega$ $(K = 4)$. The frequencies of the two-pole cases were selected around 15 GHz to illustrate the broad-bandwidth characteristics of the design. Table I shows the design parameters employed in this calculation.

Our next example illustrates the return loss as a function of f_2 for fixed $f_1 = 10$ GHz. For f_2 in the range of 14–22 GHz, $K = 2$ and 6, the return loss is presented in Figs. 3 and 4, respectively. The data shows that the design can behave like an acceptable bandpass filter/transformer for reasonably low impedance transform ratios K, becoming marginal for $K = 6$ for designs that address the fundamental and first harmonic.

TABLE I
DESIGN PARAMETERS FOR QUARTER-WAVELENGTH VERSUS TWO-POLE STUDY. THE CENTER FREQUENCY WAS CHOSEN AT 15 GHz FOR $Z_0 = 200\ \Omega$ AND $R_L = 50\ \Omega$. HERE, L DENOTES THE TOTAL LENGTH

	λ/4	Double Pole	2-pole case # 1	2-pole case # 2	2-pole case # 3
f1	15	15	13	12	10
f2	N/A	15	17	18	20
Z0	200	200	200	200	200
Z1	N/A	141	139	136	125
Z2	100	70.7	71.9	73.6	80
RL	50	50	50	50	50
K	4	4	4	4	4
L / λ1	0.25	0.5	13/30	0.4	1/3

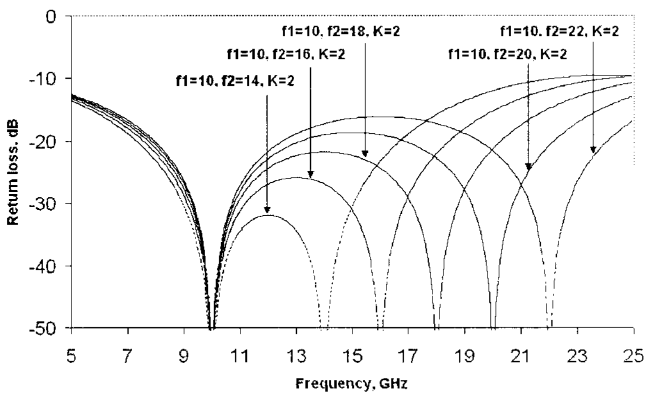

Fig. 3. Return-loss characteristics as a function of f_2 for fixed $f_1 = 10$ and $K = 2$.

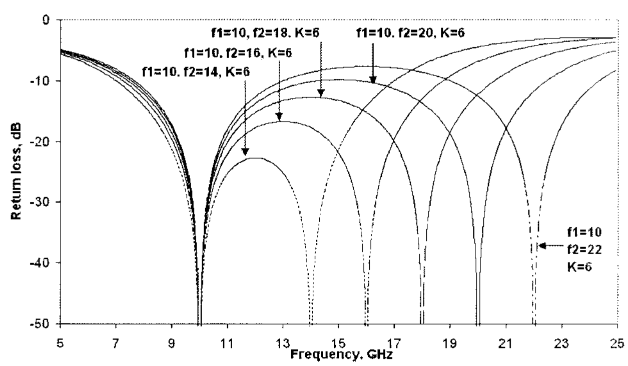

Fig. 4. Return-loss characteristics as a function of f_2 for fixed $f_1 = 10$ and $K = 6$.

The next case corresponds to the fundamental and its first harmonic. We chose $f_1 = 10$ GHz and $f_2 = 20$ GHz to illustrate the fact that the present solution with two sections is exact. Return-loss data as a function of the frequency is presented in Fig. 5 for values of K ranging from 2 to 20. The quality of the solution is evident from this figure.

Another case we consider here is the validation of the properties (31) corresponding to K and its inverse $1/K$. The frequencies chosen are $f_1 = 10$ GHz and $f_2 = 20$ GHz for $K = 2$, 4, and 20 and their inverses. Table II presents the corresponding design parameters, while Fig. 6 illustrates the return loss as a function of the frequency. The calculation clearly verifies the validity of (31).

Our final example serves to illustrate the differences between the traditional quarter-wave design, and the extreme cases of the present method, namely, the single and double poles. Here, the single pole was approximated by using $f_2 = 10{,}000\ f_1$, and for $f_1 = 15$ GHz, the three sets of data are shown in Fig. 7.

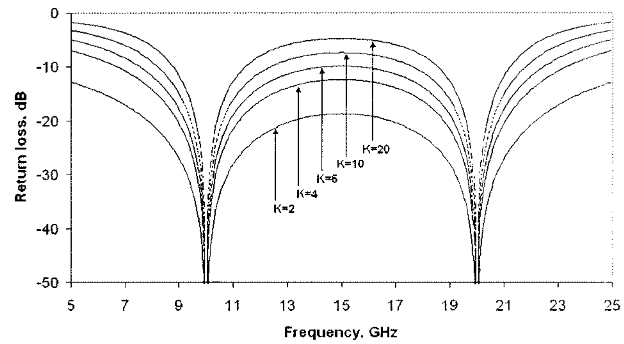

Fig. 5. Case of a fundamental and its first harmonic for variable impedance transform ratios K. $f_1 = 10$ GHz. $f_2 = 20$ GHz.

TABLE II
DESIGN PARAMETERS FOR STUDY RELATED TO PROPERTIES OF K AND ITS INVERSE $1/K$

	Case # 1	Case # 2	Case # 3	Case #4	Case # 5	Case # 6
f1	10	10	10	10	10	10
f2	20	20	20	20	20	20
Z0	25	100	200	12.5	1000	2.5
Z1	31.5	79.3	125	20	340	7.35
Z2	39.6	63.1	80	31.2	147	17
RL	50	50	50	50	50	50
K	1/2	2	4	1/4	20	1/20

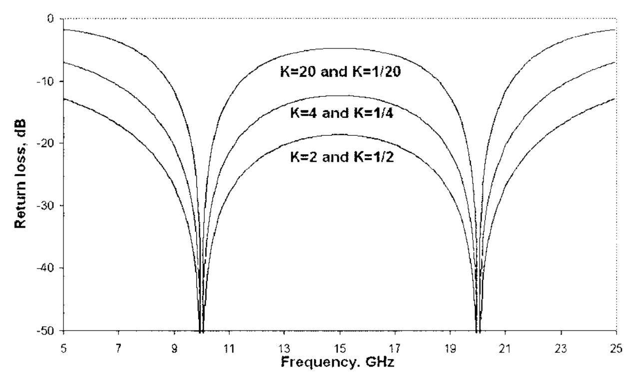

Fig. 6. Return-loss study related to properties of K and its inverse $1/K$. $f_1 = 10$ GHz, $f_2 = 20$ GHz.

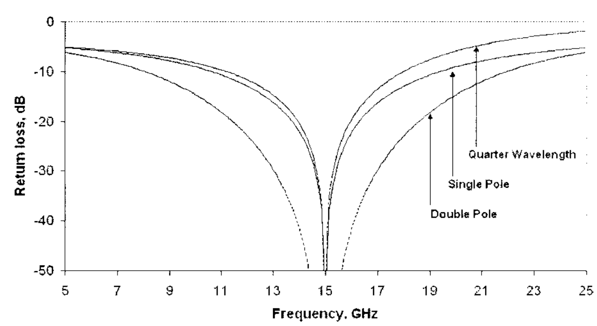

Fig. 7. Comparison of traditional quarter-wave design with exceptional single- and double-pole ($K = 4$) cases.

V. Conclusion

A novel and elementary two-section impedance transformer has been shown to be capable of dual-band operation under unrestricted load and frequency conditions. Exact closed-form design equations have been presented, and the results have been validated numerically. The case of a fundamental and its first harmonic is a special case of the solution that has been presented herein.

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